The Classification of Quadrilaterals – A Study in Definition*
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This monograph reports on an analysis of a small part of the mathematics curriculum, the definitions that are given to quadrilaterals. This kind of research, which we call micro-curricular analysis, is often undertaken informally by those who create curriculum, though rarely in as much detail as reported here. In fact, there are few avenues for publication of such research. This is the reason that we felt it appropriate as a CSMC publication.

The original intent of the study was to examine the definitions of seven special types of quadrilaterals: isosceles trapezoids, kites, parallelograms, rectangles, rhombuses, squares, and trapezoids. An eighth special type was added during the study, cyclic quadrilaterals.

Because of the interest in whether these definitions had changed over time, we wound up examining 100 high school geometry texts published in the United States from 1838 until the present, including 46 texts published since 1964. Some of these 100 books represent different editions by the same authors, and if all the definitions of the quadrilaterals were the same, we considered the two books as one. But if the defining condition of any type of quadrilateral changed, then we felt that the authors had reconsidered all the definitions, so we treated it as a separate book. This gave us 85 different geometry texts. We also looked at 8 college-level geometry textbooks designed for mathematics majors and 16 college texts designed for a course in mathematics for elementary school teachers. This summary gives some highlights of the monograph.

One motivation for this study was to examine the implications of the two definitions of trapezoid that can be found in mathematics textbooks. (1) A trapezoid is a quadrilateral with exactly one pair of parallel sides. We call this an exclusive definition since under it no parallelogram is a trapezoid. Under the exclusive definition, parallelograms are central in the hierarchy of trapezoids, as shown below.

(2) A trapezoid is a quadrilateral with at least one pair of parallel sides. The second definition means that all parallelograms are trapezoids; it is inclusive in the same way that rectangles include squares. Under the inclusive definition, trapezoids are more important than parallelograms and rectangles are special kinds of isosceles trapezoids.
Before 1930, many textbooks were influenced by the mutually exclusive hierarchy found in Euclid’s *Elements*. They include definitions of rhomboids, trapezia, and (in one case) oblongs. In these books, definitions of many figures are exclusive that today are inclusive. These figures are not found in textbooks published after that date. The change seems to be due to the recommendations of *The Reorganization of Mathematics in Secondary Education* (1923).

The variety of definitions of each of the quadrilaterals in today’s geometry textbooks is likely to surprise the reader. We found 7 essentially different definitions for rectangle, 4 of almost equal presence in recently published books; six of kite, including non-equivalent definitions (some that exclude rhombuses and some that exclude nonconvex figures); 7 of rhombus; and 8 of square. We also found that textbooks with exclusive definitions of quadrilaterals often implicitly use inclusive definitions when discussing the same quadrilaterals on the coordinate plane.

Quadrilaterals are typically defined as 4-sided polygons and led us to look at the definitions of polygon. In the 85 high school geometry textbooks we examined, we found 21 essentially different definitions of polygon, including 13 different definitions in the 46 texts we examined with copyrights since 1964. In most books, a polygon is a union (or collection) of coplanar segments, each of which intersects two others, one at each endpoint, and such that no two segments with the same endpoint are collinear. But some books allow a polygon to (1) be noncoplanar, (2) have sides that intersect not at their endpoints, (3) have consecutive sides that are collinear, or (4) that include the points in the region bounded by the segments.

* NOTE: Final draft of complete manuscript is under review for publication.
Preface
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